$\boxed{1}$ (1) Generalized Fourier Transform
(2) Intro to D-modules

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(1) 1. Classical Harmonic Analysis
G locally compact abelian group

Ex: $S^1, \mathbb{Z}, \mathbb{R}$

A (unitary) character of G is a homomorphism
$\chi : G \to U(1)$

$\hat{G} = (\{\text{characters of } G\}, \cdot)$ locally compact abelian gp.

<u>Ex 1</u>| $G = S^1 = [0, 2\pi]/\sim$
$e^{in(\sim)} : G \to U(1)$ $n \in \mathbb{Z}$
$\hat{G} = \mathbb{Z}$

<u>Ex 2</u>| $G = \mathbb{Z}$ $\chi(1) \in U(1)$ determines rep.
$\Rightarrow \hat{G} = S^1$

<u>Ex 3</u>| $G = \mathbb{R}$ $e^{itx} : \mathbb{R} \to U(1)$ $t \in \mathbb{R}$
$\hat{G} = \mathbb{R}$

Notice $\hat{\hat{G}} = G$ here.

<u>Thm</u>| Pontryagin duality:
$G \to \hat{\hat{G}}$ is an isomorphism
$g \mapsto \hat{g}$ where $\hat{g}(\chi) := \chi(g)$

$\hat{G}$ is Pontryagin dual

Observation: L^2 functions on G have a basis given by characters

Ex ① $f: S^1 \to \mathbb{C}$ $\quad f(\theta) = \sum_{n\in\mathbb{Z}} \alpha(n) e^{in\theta}$ "series"

② $f: \mathbb{Z} \to \mathbb{C}$ $\quad f(n) = \frac{1}{2\pi}\int_{S^1} F(\theta) e^{in\theta}$ "discrete time"

③ $f: \mathbb{R} \to \mathbb{C}$ $\quad F(x) = \int_{\mathbb{R}} \hat{f}(t) e^{itx}$ "transform"

Thm Plancheren $\quad L^2(G) \cong L^2(\hat{G})$

$$e^{ixt} \leftrightarrow \delta_t$$

Rmk $G = \mathbb{R}^n$, $L^2(G) \subset S'(G)$ tempered distributions

$$S'(\mathbb{R}^n) \cong S'(\widehat{\mathbb{R}^n})$$

Obs FT diagonalizes action of G on $\mathrm{Fun}(G)$

$$\begin{tikzcd} & G \times \hat{G} \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_{\hat{G}}"] & \\ G & & \hat{G} \end{tikzcd}$$

$$f(x) = \int_{\hat{G}} \hat{F}(t)\, e^{ixt}$$
$$= \int_{\hat{G}} (\pi_{\hat{G}}^* \hat{F})(t)\, \chi_t(x)\, dt$$
$$= (\pi_G)_* [\pi_{\hat{G}}^* \hat{F}\, \chi_t(x)]$$

$G \circlearrowright G$ by translation

$y \cdot x = x + y$

$(y \cdot f)(x) = f(x - y)$

$y \cdot e^{itx} = e^{it(x-y)} = e^{-iyt} e^{ixt}$

$\{e^{ixt}\}_{t \in \hat{G}}$ eigenbasis w.r.t. translations

why?

(2) Fourier-Mukai transform:

top'l cat G, $L^2(G)$

alg. cat H, $\mathrm{Fun}(H)$

$$\downarrow$$

sheaves(H)

X, Y smooth alg varieties

$QC(X)$: DG category of quasi-coherent sheaves on X

Toen <u>exercise</u> | dg algebr A, A-mod dg mod

understanding derived categories

$K \in QC(X \times Y)$

$\Phi_K^{X \times Y} : QC(Y) \to QC(X)$

$\mathcal{F} \mapsto (\pi_X)_* [\pi_Y^* \mathcal{F} \otimes K]$

<u>Thm</u> | (Orlov, Toen, Ben-Zvi-Francis-Nadler)

If X, Y reasonable (colimit preserving)

then any reasonable functor $\Phi : QC(Y) \to QC(X)$ is realized by a kernel K

$G \rightsquigarrow A$ abelian variety

(connected, projective, gp variety)

$\mathbb{C}^g/\Lambda \quad g \in \mathbb{Z}>0$

$\mu : A \times A \to A$ multiplication

$\hat{G} \xleftarrow{?} G$

$\downarrow$

$L^2(G) \to QC(G)$

A geometric character on A is a line bundle $\mathcal{L}$ on A s.t.

$$\mu^* \mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L} := \pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{L}$$

For $x, y \in A$

$$\mathcal{L}_{x+y} \cong \mathcal{L}_x \otimes \mathcal{L}_y$$

Rmk| previously $G \to U(1)$ homo
For $x \in G$ number
now $A \to BG_m$ homo
For $x \in A$ $\mathcal{L}_x$ a line

$(\{\text{geom characters}\}, \otimes)$
is an abelian variety
"dual abelian variety" $A^\vee$

$$\begin{tikzcd} & QC(A \times A^\vee) \arrow[dl] \arrow[d] \\ A & A^\vee \end{tikzcd}$$

$\exists$ universal bundle P on $A \times A^\vee$
called poincare line bundle
s.t. $P|_{(x, \mathcal{L})} = \mathcal{L}_x$

Rmk| $\underset{\text{number}}{e^{ixt}} \leftrightarrow \underset{\text{line}}{P}$

Thm (Mukai)
$\mathbb{F} : QC(A^\vee) \to QC(A)$

$e^{ixt} \leftrightarrow \delta_t$ corresponds to
$\mathcal{L} = P|_{\mathcal{L}} \leftrightarrow \mathcal{O}_{\mathcal{L}}$ skyscraper sheaf at $\mathcal{L}$

$(\pi_1)_* (\pi_2^* \mathcal{O}_{\mathcal{L}} \otimes P)$
$= (\pi_1)_* (P_{\pi_2^{-1} \mathcal{L}})$
$= \mathcal{L}$

1) $(\delta_y * f)(\cancel{0}\, x) = \int_{z \in G} \delta_y(z)\, f(x-z)$

$= f(x-y)$

2) ~~Thm~~ $(L^2(G), *) \leftrightarrow (L^2(\hat{G}), \cdot)$

$\delta_y \leftrightarrow e^{-iyt}$

$(\delta_y * (-)) \leftrightarrow (e^{-iyt}, \cdot)$

$\{e^{ixt}\}_{t \in \hat{G}}$ spectral decomposition of ~~a~~ $\mathrm{Fun}(G)$

	abelian, classical	non-abelian, cat.	plan
space of functions	$L^2(G) \simeq L^2(\hat{G})$	$D(\mathrm{Bun}_G \Sigma) \simeq ?$	Lec 1 D Lec 2 Bun_G
operators	$G \circlearrowright L^2(G)$ translation	Sat $\circlearrowright D(\mathrm{Bun}_G)$	Lec 3 for Sat_G
eigenbasis	$\{e^{ixt}\}_{t \in \hat{G}}$	$\{*\}_{* \in ?}$	Lec 4 for example

$$\begin{tikzcd} & G \times \hat{G} \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_{\hat{G}}"] & \\ G & & \hat{G} \end{tikzcd} \quad \rightarrow FT$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$\mathrm{Fun}\, Y \to \mathrm{Fun}(X)$

$f \to (\Phi_K^{Y \to X} f)(x)$

$= (\pi_X)_* (\pi_Y^* f \cdot K)$

schwartz kernel thm.

conversely any linear op. can be realized by a kernel

X, Y smooth

$\mathrm{Hom}(C_c^\infty(F), D(Y))$

$= D(X \times Y)$

Ex/ $C^\infty(X) \xrightarrow{id} C^\infty(X)$ realized by $\delta_{diag} \in D(X \times X)$

$\mathcal{F} * \mathcal{G} = \mu_*(\mathcal{F} \boxtimes \mathcal{G})$ convolution product

$(\mathcal{F} * \mathcal{G})^\vee \simeq (\mathcal{F}^\vee \otimes \mathcal{G}^\vee)$

$(QC(A), *) \simeq (QC(A^\vee), \otimes)$

Ex: $\mathcal{O}_\alpha * \mathcal{O}_\mu = \mathcal{O}_{\alpha \otimes M}$

$\mu^* \mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$ $\quad \mu: A \times A \to A$

$\Rightarrow \mu_x^* \mathcal{L} \cong \mathcal{L}_x \otimes \mathcal{L}$ $\quad \downarrow x$

$\mu_x: A \to A$

$\{\mathcal{L}\}_{\mathcal{L} \in A^\vee}$ eigenbasis w.r.t. $\underbrace{\{\mu_x\}}_{\text{translations}}$

$QC(A^\vee) \underset{\mathbb{F}(P)}{\overset{\mathbb{F}(P)}{\rightleftarrows}} QC(A)$

$\{\mathcal{O}_\mathcal{L}\}$ skyscrapers

(3) Intro to D-modules

1. D-modules on A^1

$\mathcal{O}_X$-module is module over $\mathcal{O}_X$

D-module is module over $\mathcal{D}_X$

differential operators

$\underbrace{X = \mathbb{A}^1, \quad D = D(\mathbb{A}^1) = \dfrac{\mathbb{C}\langle x, \partial\rangle}{\partial x - x\partial = 1}}_{\text{Weyl Algebra}}$

Rmk/ This is the quantum observables for Quantum Mechanics on $\mathbb{A}^1$

"Hilbert space" $= \mathbb{C}[x]$

$\mathcal{D} \circlearrowright \mathbb{C}[x]$

$x \rightsquigarrow x\cdot$

$\partial \rightsquigarrow \partial/\partial x$

Goal/ Find other $\mathcal{D}$-modules

f: function or distribution

$M_f = \mathcal{D}\cdot f = \mathcal{D}/\mathcal{D}P$ $\quad P$ is PDE for f

① Ex: $M_1 = \mathcal{D}\cdot 1 = \mathcal{D}/\mathcal{D}\partial = \mathbb{C}[x]$

② Ex. $M_{1/x} = \mathcal{D}\cdot 1/x = \mathcal{D}/\mathcal{D}(\partial x) = \mathbb{C}[x, x^{-1}]$

③ x^λ, $\lambda \in \mathbb{C}\setminus\mathbb{Z}$ $\quad \partial(x^\lambda) = \lambda x^{\lambda-1}$

$M_{x^\lambda} = \mathcal{D}x^\lambda = \mathcal{D}/\mathcal{D}(x\partial-\lambda) = \mathbb{C}[x,x^{-1}]x^\lambda$

④ δ_0, $M_{\delta_0} = \mathcal{D}\delta_0 = \mathcal{D}/\mathcal{D}x = \mathbb{C}[\partial]$

⑤ $M_{e^{\lambda x}} = \mathcal{D}e^{\lambda x} = \mathcal{D}/\mathcal{D}(\partial-\lambda) = \mathbb{C}[x]e^{\lambda x}$

Note

$$0 \to M_1 \to M_{1/x} \to M_{\delta_0} \to 0$$

Rmk/ In alg. geometry, D-module captures generalized functions.

e.g. $M_{e^{\lambda x}}$ D-module

consider $\mathrm{Hom}_D(M_{e^{\lambda x}}, \mathcal{O})$ = solutions to $PF=0$ in $F=\mathcal{O}$ $= \mathbb{C}e^{\lambda x}$

Claim/ D is almost-commutative

$$D^{\hbar} = \frac{\langle x, \partial \rangle}{\partial x - x\partial - \hbar} = \begin{cases} D, & \hbar \neq 0 \\ \mathbb{C}[x,y] = \mathcal{O}(T^*\mathbb{A}^1), & \hbar = 0 \end{cases}$$

Filtration on D:

$$D_{\leq n} = \{ \dots \partial^{\leq n} \}$$

$$\mathrm{gr}\, D = \bigoplus_n D_{\leq n}/D_{\leq n-1} = \mathcal{O}(T^*\mathbb{A}^1)$$

D-module M might admit a Filtration

$$D_{\leq n} \cdot M_{\leq m} \subseteq M_{m+n}$$

$$\mathrm{gr}\, D \circlearrowright \mathrm{gr}\, M$$

Defn/ Singular support of M (D-mod on $\mathbb{A}^1$) is support of gr M as a module over gr D

$$SS(M) \subset T^*\mathbb{A}^1$$

Ex/

(1) $M_1 = D/D\partial = \mathbb{C}[x]$

$\mathbb{C}[x,y] \circlearrowleft \mathbb{C}[x] \rightsquigarrow \mathbb{A}^1 \subset T^*\mathbb{A}^1$

$\mathbb{C}[x,y] = \mathrm{gr}\, D \qquad \mathbb{C}[x] = \mathrm{gr}\, M = m$

(2) $M_{1/x} = D/D(\partial x) = \mathbb{C}[x,x^{-1}] \rightsquigarrow \mathbb{A}^1 \cup T_0^*\mathbb{A}^1$

$D/DP \quad \sigma(P)$ symbol

$SS(M) = \{\sigma(P) = 0\}$

$P = \partial x$ recipe: For given P

$P = xy$ take only highest

order in ∂ and change

∂ to y

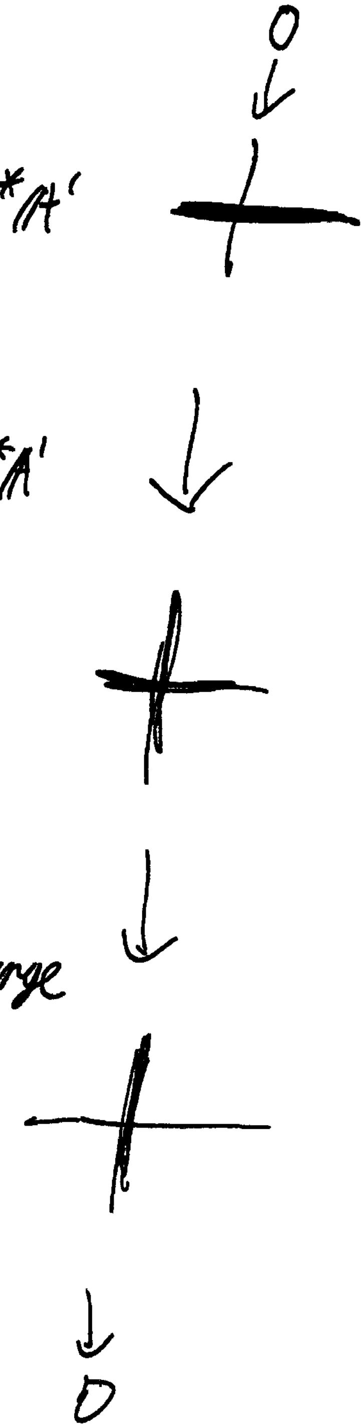
(3) $M_{\delta_0} = D/Dx = \mathbb{C}[y]$

$P = x$

(4) $M_{x^\lambda} = D/D(x\partial - \lambda)$

$P = x\partial - \lambda$

$\sigma(P) = xy$



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